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Piecewise Monotone Spline Interpolation

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Let (x_i, y_i), $i = 0, 1,..., k$, be a set of points, with $x_0 < x_1 < \cdots < x_k$. We prove the existence of a spline function of specified deficiency, $f(x)$, which satisfies $f(x_i) = y_i$, $i = 0, 1,..., k$, and which is monotone on each of the intervals $[x_{i-1}, x_i]$, $i = 1, 2,..., k$.

Let (x_i, y_i), $i = 0, 1,..., k$, be a set of points in the plane, with $x_0 < x_1 < \cdots < x_k$. It was shown independently by Wolibner [4] and Young [5] that if $y_{i-1} \neq y_i$, $i = 1, 2,..., k$, then there exists a polynomial $p(x)$ such that $p(x_i) = y_i$, $i = 0, 1,..., k$, and p is monotone on each of the intervals $[x_{i-1}, x_i]$, $i = 1, 2,..., k$. Both proofs, however, fail to give any information as to the degree of the polynomial needed. Rubinstein [3] has obtained such a result, but only in a very restricted case, i.e., when $k = 2$ and $y_0 < y_1 < y_2$. In this note we consider the problem of piecewise monotone interpolation (PMI), but use splines as our interpolating functions.

DEFINITION. Let $S_n^j = S_n^j(x_0, x_1, ..., x_k)$, $0 \leqslant j \leqslant n - 1$, be the set of all functions $f \in C^j[x_0, x_k]$ and such that f agrees with a polynomial of degree $\leqslant n$ on $[x_{i-1}, x_i]$, $i = 1, 2,..., k$. $f \in S_n^j$ is said to be a spline of order n with deficiency $n - j$. $f \in S_n = S_n^{n-1}$ is called a simple spline of order n (cf., Ahlberg, Nilson, and Walsh [1, p. 7]).

Since S_1 consists of all continuous piecewise linear functions, it is clear that PMI is always possible with functions from S_1. For S_n, $n \geqslant 2$, however, the possibility of PMI will depend upon the data. On the other hand, for splines of certain specified deficiencies, we have the following result:

THEOREM. *Let $x_0 < x_1 < \cdots < x_k$ and let y_i, $i = 0, 1,..., k$, be arbitrary. Then, for each n, there exists a unique $f \in S_{2n+1}^n$ such that $f(x_i) = y_i$, $i = 0, 1,..., k$, and f is monotone on each of the intervals $[x_{i-1}, x_i]$, $i = 1, 2,..., k$.*

Proof. On $[x_{i-1}, x_i]$, $i = 1, 2,..., k$, let $p_i(x)$ be the unique polynomial of degree $\leqslant 2n + 1$ which satisfies the Hermite interpolation problem,

$p_i(x_{i-1}) = y_{i-1}$, $p_i(x_i) = y_i$, $p_i^{(j)}(x_{i-1}) = p_i^{(j)}(x_i) = 0$, $j = 1, 2,..., n$ (cf., Natanson [2, p. 15]). If $y_{i-1} = y_i$, then $p_i(x) \equiv y_i$. If $y_{i-1} \neq y_i$, then consider $p_i'(x)$, which is a polynomial of degree $\leqslant 2n$. Since p_i' has zeros of multiplicity n at x_{i-1} and x_i, it can have no other zeros. Thus p_i' is of constant sign on (x_{i-1}, x_i), so that p_i is strictly monotone on $[x_{i-1}, x_i]$. Now let $f(x) = p_i(x)$ for $x \in [x_{i-1}, x_i]$, $i = 1, 2,..., k$. Then $f(x_i) = y_i$, $i = 0, 1,..., k$, $f \in S_{2n+1}^n$, and f is monotone on $[x_{i-1}, x_i]$, $i = 1, 2,..., k$.

Remark 1. The method of the theorem can be used to show that for each n there exists $f \in S_{2n}^{n-1}$ which solves the PMI problem. In this case, however, the interpolating spline is not unique.

Remark 2. The theorem is best possible in the sense that S_{2n+1}^n are the splines of minimal deficiency for which PMI is always possible. For consider $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$; $y_0 = y_1 = 0$, $y_2 = y_3 = 1$. Let $f \in S_{2n+1}^j$, where $j \geqslant n + 1$, and suppose that f interpolates the data piecewise monotonely. Then $f \equiv 0$ for $x \in [-1, 0]$ and $f \equiv 1$ for $x \in [1, 2]$. f must also satisfy the following:

(a) $f(0) = f'(0) = \cdots = f^{(j)}(0) = f'(1) = f''(1) = \cdots = f^{(j)}(1) = 0$,

(b) $f(1) = 1$, and

(c) f agrees with a polynomial p of degree $\leqslant 2n + 1$ on $[0, 1]$. Hence p' is a polynomial of degree $\leqslant 2n$ having at least $2j - 1 \geqslant 2n + 1$ zeros. Therefore $p' \equiv 0$, so that $p(x)$ is a constant. But $p(0) = 0$ and $p(1) = 1$, so that no such $f \in S_{2n+1}^j$ exists.

References

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